On the Construction of Generalized Grassmann Coherent States

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Abstract

A generalized definition of a deformation of the fermionic oscillator (k-fermionic oscillators) is proposed. Two prescriptions for the construction of generalized Grassmann coherent states for this kind of oscillators are derived. The two prescriptions differs in the nature of the generalized Grassmann variables used. While we use Majid's definition for such variables in the first case, Kerner's definition is used in the second one.

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1   Introduction

The fermionic oscillator is generated by the operator of creation and annihilation obeying the following anticommutation relations:

$$
\begin{array}{rcl}
\{a, a^+\} & = & a\, a^+ + a^+\, a = I \\
\{N, a\} & = & a \quad ; \quad \{N, a^+\} = a^+ \quad \text{with } N = a^+\, a
\end{array}
\tag{1}
$$

and a nilpotency condition:

$$
(a)^2 = (a^+)^2 = 0
\tag{2}
$$

which is the algebraic formulation of Pauli's exclusion principle for fermions.

At the algebraic level, a deformation of the harmonic oscillator is an alteration of the (anti-)commutation relations between the different operators. Generally it consists in introducing a set of new dimensionless parameters, which by taking some special values enable to recover the usual oscillator (fermionic or bosonic). In this spirit, several deformations appeared in the literature, either depending on one parameter [1, 2, 3] or two parameters [4].

However all this deformations have some common properties which made it easy to unify all this deformations into a single generalized deformation [5, 6, 7, 8]. This unification is mainly done at the level of the Fock representation and is totally encrypted into a function of the particle number. By Choosing some special forms for this function one recovers a particular deformation of the harmonic oscillator. A similar generalization (unification) of the fermionic deformations [9] was proposed in [10] where the deformation is achieved using some function of the number operator with some extra conditions compared to the bososnic deformations).

However this paper we are going to use a similar generalization to that presented in [8], with some extra conditions again to adapt it to a fermionic deformation. In fact a fermionic deformation is also characterized by a degree of nilpotency of the different operators[2]. It is worth mentioning that this nilpotency is sometimes referred to as a generalized Pauli exclusion principle. Which in Fock space representation reflects into the finite dimensionality of this space.

It is because of this finite dimensionality that Grassmann variables are needed in order to construct the relevant coherent states [11].

It is known [11, 12] that in order to construct fermionic coherent states one needs to use Grassmann variables to take account of the nilpotency of the fermionic operators. It is clear then that in order to construct coherent states related to deformations of the fermionic oscillators and to generalizations of these deformations one should use some generalized Grassmann variables.

Kerner [13] and Majid [14] Introduced such variables that are characterized in one hand by non trivial commutation relations and on the other hand by a degree of nilpotency higher than the degree of nilpotency of the usual Grassmann variables.

This paper is devoted to the construction of Generalized Grassmann Coherent States GGCS, these are states related to a generalized (unified) deformation of the fermionic oscillator by using the generalized Grassmann variables. It is organized as follows:

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[2]generalizing the nilpotency conditions (2)

In the second section, inspired from the results of [8], a generalization of the fermionic deformations is proposed.

The third section is devoted to the construction of the Generalized Grassmann Coherent States, first using Majid's definition of generalized Grassmann variables (subsection 3.1) then using Kerner's definition of the Z_3-graded Grassmann variables.

The different results are summarized and discussed in the last section.

2   Deformation of the Fermionic Oscillator

A deformation of the fermionic oscillator, like a bosonic deformation, consists in introducing some parameter(s) which for some special values allow the recovering of the fermionic oscillator. In addition to this it is characterized by a nilpotency degree, $k \geq 2$, of the creation and annihilation operators. This last property generalizes Pauli's exclusion principles by authorizing multiparticle states.

After a thorough study of the different ways to unify deformations of the bosonic oscillator [5, 6, 7, 8], we find that the one which may suit a unification of deformations of the fermionic oscillator is the one presented in [8]. Namely, a generalized deformed fermionic algebra, $\mathcal{F}_q$, is freely generated by operators of annihilation, creation and identity $\{a, a^+, I\}$ with "q"-commutation relations:

$$
\begin{aligned}
[a, a^+]_q &= aa^+ - q\, a^+ a &= \Delta' \\
[a, \Delta]_q &= a\, \Delta - q\, \Delta\, a &= \Delta'\, a \\
[\Delta, a^+]_q &= \Delta\, a^+ - q\, a^+ \Delta &= a^+\, \Delta' \\
&\;\;\vdots
\end{aligned}
\tag{3}
$$

where $\Delta = a^+\, a$ and q is a complex parameter. Δ' is generally interpreted as a q-derivative of Δ and is commuting with it.

Unlike a bosonic oscillator, a deformed fermionic oscillator is characterized (in addition to the commutation relations) by a degree of nilpotency of the operators of creation and annihilation:

$$
(a)^k = \left(a^+\right)^k = 0 \, .
\tag{4}
$$

Generally, the degree of nilpotency k depends on the value of the parameter(s) of deformation.

It is easy to check at this stage that one can recover the ordinary (non-deformed) fermionic oscillator (1, 2) from the algebra above by choosing $q = -1$, $\Delta = N$ and $\Delta' = I$ in (3) and $k = 2$ in (4). However, in order to obtain the different deformations of the fermionic oscillator, it is convenient to work in the Fock space representation of the algebra $\mathcal{F}_q$. In the following we will construct this representation space.

If we were to consider only (3), then the Fock representation would be the one given in [8]; i.e., the Fock basis vectors $\{|n\rangle\}$, $n = 0, 1, 2 \cdots$, eigenvectors of the number operator, are constructed from the vacuum state, $|0\rangle$ $(a|0\rangle = 0)$, by

successive actions of the creation operator:

$$|n\rangle = \frac{(a^+)^n}{(\rho_n)^{1/2}}|0\rangle \tag{5}$$

where we have introduced the function ρ_n which is defined through the action of the different operators on the basis vectors:

$$\begin{aligned}
a\,|n\rangle &= (\rho_n)^{1/2}|n-1\rangle \\
a^+\,|n\rangle &= (\rho_{n+1})^{1/2}|n+1\rangle \\
\Delta\,|n\rangle &= \rho_n\,|n\rangle \\
\Delta'\,|n\rangle &= (\rho_{n+1} - q\,\rho_n)\,|n\rangle
\end{aligned} \tag{6}$$

It is clear that in order to take account of the nilpotency conditions (4) some further conditions should be imposed on the function ρ_n. In fact, the highest number state one should be allowed to construct is $|k-1\rangle$ this means that the function ρ_n should be defined such that:

$$\rho_k = 0, \tag{7}$$

k being the nilpotency degree of the fermionic operators (4).

In this way, we recover the main difference between bosonic oscillators (deformed or non-deformed) and fermionic ones: Pauli's exclusion principle. Notice that, while at the algebra level the generalized exclusion principle is implemented through the nilpotency of the operators (4), at the representation level it is present through the finite dimensionality of the Fock representation (this is in contrast to the infinite dimension in the case of the bosonic deformation).

The algebra $\mathcal{F}_q$ defining a generalized fermionic oscillator, one should be able to recover all the deformations of the fermions oscillator as particular choices of the function ρ_n. In the following we discuss some of the most famous cases.

- Ordinary Fermions: In this case $k = 2$, and $q = -1$. Then choosing $\rho_n = n$, $n = 0, 1$. one obtains the two-dimensional Fock representation:

$$\begin{aligned}
a\,|0\rangle &= 0 \;; & a\,|1\rangle &= |0\rangle \\
a^+\,|0\rangle &= |1\rangle \;; & a^+\,|1\rangle &= 0
\end{aligned} \tag{8}$$

- Arik-Coon q-oscillator [1]:
  This oscillator is defined by the following commutation relations:

$$[a, a^+]_q = aa^+ - qa^+a = I. \tag{9}$$

When the complex parameter of deformation q is a root of unity, i.e. $q^k = 1$; $q = \exp i\frac{2\pi}{k}$ (k being an integer) one obtains nilpotency of the operators a and a^+ as in (4). This case is also a particular case of $\mathcal{F}_q$, as it is obtained from it by choosing

$$\rho_n = \{n\}_q = \frac{1 - q^n}{1 - q} \tag{10}$$

and it is easy to check that $\rho_k = 0$.

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- Biedenharn-Macfarlane q-oscillator [2]: It is defined through the q-commutation relations:

$$aa^+ - qa^+a = q^{-N}. \tag{11}$$

Here also when the parameter of deformation is a root of unity, $q^k = 1$ one obtains nilpotency of the operators a and a^+. It is recovered from $\mathcal{F}_q$ if one puts

$$\rho_n = [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{12}$$

and here also $\rho_k = [k]_q = 0$ because $q^k = 1$.

- Chung and Parathasarathy [15]:
  This deformation is based on the following q-deformed equation:

$$aa^+ + q' \, a^+a = q'^N \ , \tag{13}$$

it is recovered by putting $q' = -q$ in (3) and choosing:

$$\rho_n = q^{n-1} \sin^2\left(\frac{n\pi}{k}\right) \ . \tag{14}$$

One can check that, independently of the values of q, $\rho_k = 0$, which means that $a^k = (a^+)^k = 0$.

3 Generalized Grassmann Coherent States: GGCS

3.1 1st Case: using Majid's variables.

According to Majid [14] a Z_k-graded Grassmann algebra is generated by the variables obeying the following relations:

$$\begin{aligned}
\theta_i\theta_j &= q \, \theta_j\theta_i \\
(\theta_i)^k &= 0
\end{aligned} \qquad i, j = 1, 2, \ldots \quad i < j \tag{15}$$

where q is a kth root of unity, i.e. $q = e^{\frac{2\pi i}{k}}$:

$$q^k = 1 \ , \quad \bar{q} = q^{-1} = q^{k-1} \ , \quad 1 + q + \ldots q^{k-1} = 0 \ . \tag{16}$$

One also introduces hermitian conjugate variables $\bar{\theta} = \theta^\dagger$:

$$\begin{aligned}
\bar{\theta}_i\bar{\theta}_j &= q\bar{\theta}_j\bar{\theta}_i \\
(\bar{\theta}_i)^k &= 0 \ .
\end{aligned} \qquad i < j \tag{17}$$

and we have the following commutation relations between the two sectors:

$$\begin{aligned}
\theta_i\bar{\theta}_j &= \bar{q}\bar{\theta}_j\theta_i \\
\bar{\theta}_i\theta_j &= \bar{q}\theta_j\bar{\theta}_i \ .
\end{aligned} \qquad i < j \tag{18}$$

All these relations can be written in a compact form as follows

$$\alpha_i \beta_j = q^{ab} \beta_j \alpha_i \quad i < j \tag{19}$$

where α and β stand for θ and/or $\bar{\theta}$, while a and b respectively denote the grades of α and β; where to the $\theta's$ we attribute a grade 1 and a grade $k-1$ to the $\bar{\theta}$'s.

In the following, and since we are dealing with a one mode oscillator, we will be mainly interested in the one dimensional case, which means that we will omit the subscripts.

We also have the following integration rules generalizing Berezin's rules of integration:

$$\int d\alpha \; \alpha^n = \delta_{n,k-1} \; . \tag{20}$$

where $\alpha = \theta$, $\bar{\theta}$ and n is any positive integer. And we have the following relations

$$
\begin{array}{rclcrcl}
\theta d\bar{\theta} &=& q \; d\bar{\theta}\theta & \qquad & \bar{\theta}d\theta &=& q \; d\theta\bar{\theta} \\
\theta d\theta &=& \bar{q} \; d\theta\theta & & \bar{\theta}d\bar{\theta} &=& \bar{q} \; d\bar{\theta}\bar{\theta} \\
d\theta d\bar{\theta} &=& \bar{q} \; d\bar{\theta}d\theta \; .
\end{array}
\tag{21}
$$

These rules allow to compute the integral of any function over the Grassmann algebra written as a finite series in θ and $\bar{\theta}$:

$$f(\theta,\bar{\theta}) = \sum_{i,j=0}^{k-1} C_{i,j}\theta^i\bar{\theta}^j \; . \tag{22}$$

Now in order to proceed further one need to define the behaviour of these variables with respect to the oscillator operators i.e. with respect to the creation and annihilation operators. In order to do this one should be inspired by the (usual) fermionic case where the Grassmann variables not only anticommute with each other but anticommute also with the fermionic creation and annihilation operators. Thus one should adopt general commutation relations between the Generalized Grassmann variables and the Generalized fermionic operators such that one recovers the anticommutativity when ($k = 2$):

$$
\begin{array}{rclcrcl}
\theta a^+ &=& q \; a^+\theta & \qquad & \theta a &=& \bar{q} \; a\theta \\
\bar{\theta}a^+ &=& \bar{q} \; a^+\bar{\theta} & & \bar{\theta}a &=& q \; a\bar{\theta} \; .
\end{array}
\tag{23}
$$

Now we have all the ingredients to construct Generalized Coherent States associated to the Generalized fermionic oscillator $\mathcal{F}_q$.

Coherent States (the canonical ones) have many properties and many of these properties can be considered as defining properties for general coherent states [11]. And the defining property for coherent states associated to oscillators is that they are eigenstates of the annihilation operator with the eigenvalue given by the label of the coherent states. So in our case we must find states $|\theta\rangle_k$ such that

$$a|\theta\rangle_k = \theta|\theta\rangle_k \; . \tag{24}$$

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The states are written in a standard form in the Fock basis with general coefficients:

$$|\theta\rangle_k = \sum_{n=0}^{k-1} \alpha_n \theta^n |n\rangle \tag{25}$$

Now to find the states $|\theta\rangle_k$ one should find the coefficients α_n for which the property (24) is verified.

$$
\begin{aligned}
a|\theta\rangle_k &= \sum_{n=0}^{k-1} \alpha_n a \theta^n |n\rangle \\
&= \sum_{n=0}^{k-1} \alpha_n q^n \theta^n a |n\rangle \\
&= \sum_{n=1}^{k-1} \alpha_n q^n \theta^n (\rho_n)^{\frac{1}{2}} |n-1\rangle \\
&= \theta \sum_{n=0}^{k-2} \alpha_{n+1} q^{n+1} \theta^n (\rho_{n+1})^{\frac{1}{2}} |n\rangle
\end{aligned}
$$

which by definition should equal

$$\theta|\theta\rangle_k = \theta \sum_{n=0}^{k-1} \alpha_n \theta^n |n\rangle$$

So the condition on α_n for the equation (24) to be verified is:

$$\alpha_n = \alpha_{n+1} \, q^{n+1} \, (\rho_{n+1})^{\frac{1}{2}} \; . \tag{26}$$

Iteratively solving this equality yields the solution:

$$\alpha_n = \frac{1}{(\rho_{n+1})^{\frac{1}{2}}} \bar{q}^{\frac{n(n+1)}{2}} \tag{27}$$

Then the GGCS associated to $\mathcal{F}_q$ are written in the Fock basis as:

$$|\theta\rangle_k = \sum_{n=0}^{k-1} \bar{q}^{\frac{n(n+1)}{2}} \frac{\theta^n}{(\rho_n!)^{\frac{1}{2}}} |n\rangle \; . \tag{28}$$

Using equation (5) one can see that the states $|\theta\rangle_k$ are generated from the vacuum through:

$$|\theta\rangle_k = \sum_{n=0}^{k-1} \bar{q}^{\frac{n(n+1)}{2}} \frac{\theta^n (a^+)^n}{(\rho_n!)} |0\rangle \; . \tag{29}$$

Using equation (23)one can prove that:

$$\theta^n (a^+)^n = q^{\frac{n(n+1)}{2}} (a^+\theta)^n \tag{30}$$

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which permit us to rewrite the GGCS in a more conventional way:

$$|\theta\rangle_k = \sum_{n=0}^{k-1} \frac{(a^+\theta)^n}{\rho_n!} \, |0\rangle \ := \ \exp_q(a^+\theta)\,|0\rangle \ . \tag{31}$$

where the generalized deformed exponential function introduced is defined through:

$$\exp_q(x) = \sum_{n=0} \frac{x^n}{\rho_n!} \tag{32}$$

Next we will prove that the GGCS, as all coherent states, do provide a resolution of unity [11]. We will look for a resolution of unity in the form:

$$\iint d\bar{\theta}d\theta \ \omega(\bar{\theta}\theta) \ |\theta\rangle_{kk}\langle\theta| = I \ , \tag{33}$$

where the weight function is written as follows

$$\omega(\bar{\theta}\theta) = \sum_{n=0}^{k-1} c_n \ \theta^n\bar{\theta}^n \ . \tag{34}$$

Then the problem of proving the resolution of unity reduces to finding the adequate weight function, that is, to finding the coefficients c_n such that (33) holds:

By replacing (34) and (28), the LHS of (33) can be written as follows:

$$\iint \sum_{l,n,p=0}^{k-1} d\bar{\theta}d\theta \ c_n\theta^n\bar{\theta}^n \ \bar{q}^{\frac{l(l+1)}{2}} q^{\frac{p(p+1)}{2}} \ \frac{\theta^l\bar{\theta}^p}{(\rho_l!)^{\frac{1}{2}}(\bar{\rho}_p!)^{\frac{1}{2}}} \ |l\rangle\langle p| \ . \tag{35}$$

Taking account of the integration rules and the q-commutation rules it becomes:

$$\iint d\bar{\theta}d\theta \ \sum_{l,n=0}^{k-1} \frac{c_n q^{nl}}{|\rho_l|!} \ \theta^{n+l}\bar{\theta}^{n+l}|l\rangle\langle l| \ . \tag{36}$$

which in turn by using the completeness of the Fock space basis $\sum_{l=1}^{k-1}|l\rangle\langle l| = I$

becomes:

$$I = \sum_{\substack{n,l=0 \\ n+l=k-1}}^{k-1} \frac{c_n q^{nl}}{|\rho_l|!} \iint d\bar{\theta}d\theta\theta^{n+l}\bar{\theta}^{n+l} \ |l\rangle\langle l| \ . \tag{37}$$

Since this should give the identity operator, one has the following constraint on the coefficients c_n:

$$\frac{c_n q^{nl}}{|\rho_l|!} = 1 \quad \text{and} \quad n+l = k-1 \ . \tag{38}$$

i.e.

$$\begin{aligned} c_n &= \bar{q}^{n(k-n-1)}|\rho_{k-n-1}|! \\ &= q^{n(n+1)}|\rho_{k-n-1}|! \end{aligned} \tag{39}$$

The weight function appearing in the resolution of unity (33) is therefore given by

$$\omega(\bar{\xi}\xi) = \sum_{n=0}^{k-1} q^{\frac{n(n+1)}{2}}|\rho_{k-n-1}|!(\bar{\xi}\xi)^n \ . \tag{40}$$

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3.2 1st Case: using Kerner's variables.

Kerner's starting point for generalizing the Grassmann variables is somehow the same as Majid's. In fact, both generalizations start from the fact that the Grassmann algebra is a Z_2-graded algebra. Then suggesting that a generalized Grassmann algebra should be a Z_k-graded one, where k is a positive inee=ger $k \geq 2$. However the procedures diverge in the way of implementing this Z_k-graded structure on the constructed algebra. In fact, Majid implement this by replacing -1 (the second root of unity) by a $q = \exp\frac{2pii}{k}$ (a kth root of unity). Thus changing the Grassmann anticommutation relation by a q-commutation relation (19). While Kerner's point of view is fundamentally different [13]:

The Z_2-graded structure of the Grassmann algebra reflects itself in the fact that Z_2 (the cyclic group) is a symmetry group for this algebra; that is for any two elements of the Grassmann algebra the following property holds

$$\mathcal{Z}_2(\xi_i\xi_j) = \xi_i\xi_j + \xi_j\xi_i = 0 \quad \text{where} \quad \mathcal{Z}_2 \text{ is an element of } Z_2 \tag{41}$$

(from which, one deduces the anticomutativity property of the Grassmann variables and their nilpotency).

A generalized Z_k-graded Grassmann algebra then should generalize this property by imposing a symmetry with respect to the cyclic group Z_k. Kerner considered (and totally solved) the case where $k = 3$. In this case the Z_3-graded Grassmann variables obey the following ternary property (symmetry with respect to the cyclic group Z_3):

$$\mathcal{Z}_3(\xi_i\xi_j\xi_k) = \xi_i\xi_j\xi_k + \xi_j\xi_k\xi_i + \xi_k\xi_i\xi_j = 0 \qquad \mathcal{Z}_3 \text{ being an element of } Z_3 \tag{42}$$

A particular solution of the constraint (42) is given by the ternary relation:

$$\xi_i\xi_j\xi_k = j\ \xi_j\xi_k\xi_i = j^2\ \xi_k\xi_i\xi_j \tag{43}$$

i.e. under a cyclic permutation of the elements of a ternary product a factor j appears where j is a cubic root of unity:

$$j = \exp\left(\frac{2\pi i}{3}\right)\ ;\ j^3 = 1\ ;\ \bar{j} = j^2\ ;\ j^2 + j + 1 = 0 \tag{44}$$

So the Z_3-graded Grassmann variables is obtained by assuming that there are no binary relations among these variables (i.e. products of the form $\xi_1\xi_2$ and $\xi_2\xi_1$ are considered as independent elements). Instead of this, ternary relations (43) are given.

Moreover, two important properties follow automatically from (43):

- $(\xi_i)^3 = 0$

- $\xi_i\xi_j\xi_k\xi_l = 0$

One also introduces grade-2 elements [13], which are duals to the ξ's and obey similar relations with j replaced by j^2:

$$\bar{\xi}_i\bar{\xi}_j\bar{\xi}_k = q^2\ \bar{\xi}_j\bar{\xi}_k\bar{\xi}_i = q\ \bar{\xi}_k\bar{\xi}_i\bar{\xi}_j \tag{45}$$

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There exist binary relations but only for a product involving a grade-1 element and a grade-2 one:

$$\xi_i \bar{\xi}_j = q \, \bar{\xi}_j \xi_i \tag{46}$$

In the following, and as in the previous section, we will omit the subscripts as we will be dealing mainly with the one mode oscillator and a one dimensional Grassmann algebra.

Integration over this algebra is carried with the same relations (20). In this case ($k = 3$), these are written explicitly:

$$\int d\xi.1 = \int d\bar{\xi}.1 = \int d\xi.\xi = \int d\bar{\xi}.\bar{\xi} = 0 \tag{47}$$
$$\int d\xi.\xi^2 = \int d\bar{\xi}.\bar{\xi}^2 = 1$$

and here also, the rules allows to compute the integral of any function over the Grassmann algebra.

In order to proceed further one need to define the behavior of these variables with respect to the generalized fermionic (creation and annihilation) operators, i.e. relations similar to (23). To write these relations, one should first note that there is a deep difference between the two generalized Grassmann algebras discussed. In fact, grading is the main rule governing the relations in Kerner's definition of the generalized Grassmann variables. So in order to write relations similar to (23) in this case one has to introduce a grading over the generalized fermionic oscillator algebra (3, 4). A way of doing this, is to impose that the coherent states to be constructed should be grade-0 [8, 15]. Then the grading that follows from this requirement is that a is a grade-1 while a^+ is a grade-2 element.

Combining this fact with what was announced previously leads to the conclusion that no binary relations can be imposed on ξ and a (or $\bar{\xi}$ and a^+), because no binary relations exists between elements with the same grade. These relations are however allowed between elements with different gradings:

$$\xi\, a^+ = j\, a^+ \xi \quad , \quad \bar{\xi}\, a = j^2 \, a\, \bar{\xi} \tag{48}$$

It is worth mentioning that since we restricted the definition of the generalized Grassmann variables to the Z_3-graded case ($\xi^3 = 0$), the coherent states we shall construct are associated to the generalized fermionic oscillator (3, 4) with $k = 3$ i.e. ($a^3 = (a^+)^3 = 0$); the nilpotency of the generalized Grassman variables is intimately related to the nilpotency of the fermionic operators, i.e. to the generalized Pauli's exclusion principle.

Z_3-graded Coherent States

The coherent states to be constructed should be constructed as eigenstates of the annihilation operator:

$$a|\xi\rangle_3 = \xi\, |\xi\rangle_3 \, . \tag{49}$$

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Proceeding as in the previous section, by first writing $|\xi\rangle_3$ in its general form in the representation space of the generalized fermionic oscillator algebra[3]:

$$|\xi\rangle_3 = \alpha_0|0\rangle + \alpha_1\,\xi\,|1\rangle + \alpha_2\,\xi^2|2\rangle\,, \tag{50}$$

with general coefficients α_0, α_1 and α_2. Then imposing (49), on this state and using (48) permits to determine the coefficients, and the result is:

$$|\xi\rangle_3 = |0\rangle + j^2\xi\,|1\rangle + \rho_2^{-1/2}\,\xi^2\,|2\rangle\,. \tag{51}$$

It is important that, (despite the differences in the definition of the Grassmann variables and in the relations (23) and (48)) if we replace θ by ξ in (28) and $k = 3$, (28) reduces to equation (51).

The similarity can be pushed further, as one can rewrite the GGCS (51) in the form:

$$
\begin{aligned}
|\xi\rangle_3 &= |0\rangle + a^+\xi\,|0\rangle + \rho_2^{-1}\,a^+\xi a^+\xi\,|0\rangle \\
&:= \exp_j(a^+\xi)\,|0\rangle
\end{aligned}
\tag{52}
$$

where the generalized exponential function is the same as the one in (32) with $k = 3$ or $q = j$.

To conclude the construction of the GGCS in this case, one should construct a resolution of unity in terms of these states.

Here again one follows the same method as in the previous subsection. That is,we look for a resolution of unity in the form:

$$\iint d\bar{\xi}d\xi\,\omega(\bar{\xi}\xi)\,|\xi\rangle_{33}\langle\xi| = I\,; \tag{53}$$

where $\omega(\bar{\xi}\xi) = c_0 + c_1\,\bar{\xi}\xi + c_2\,\bar{\xi}\xi\bar{\xi}\xi$. Then One has to determine the coefficients c_0, c_1 and c_2 such that the equality (53) holds.

We use the form (51) of the GGCS, the integration rules and the relations (46) and (48) and the completness of the Fock basis (6) with $k = 3$ in this case:

$$|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| = I\,. \tag{54}$$

And the result is the following:

$$\omega(\xi\bar{\xi}) = -q + \bar{\xi}\xi + \bar{\xi}\xi\bar{\xi}\xi\,. \tag{55}$$

4   Discussions

Generalized coherent states associated with generalized harmonic oscillators are defined as eigenstates of the corresponding annihilation operator. But unlike the Bosonic deformations, where the representation space is infinite dimensional, the representation space of the fermionic deformations is finite, expressing thus, a generalized Pauli exclusion principle. Because of this fact ordinary variables are not

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[3]note that the representation space (Fock space) in this case is three dimensional

suitable for the construction of the generalized coherent states. Instead of this one has to use Grassmann variables to take account of this fact.

We have shown in this paper how to construct GGCS associated to a generalized deformation of the fermionic oscillator by using the two generalizations existing in the literature of Grassmann Grassmann variables: Majid and Kerner's generalized Grassmann variables.

The result that the form of the GGCS constructed is the same:

$$|\tau\rangle = \exp_q(a^+\tau)|0\rangle \qquad (56)$$

where $\tau = \theta$ or ξ. The form which is also similar to that of the generalized coherent states associated to the bosonic deformation of the harmonic oscillator [8].

It is worth noting that the construction using Kerner's Z_3-graded Grassmann variables was carried out for the generalized fermionic oscillator (3, 4) for the case $k = 3$, i.e. $q = \exp\left(\frac{2\pi i}{3}\right)$. It is obvious that in order to generalize this construction to arbitrarily k, one has to define Z_k-graded Grassmann variables in the spirit of kerner's definition. This is done by imposong a Z_k-symmetry over the generalized Grassmann algebra to be constructed:

$$\mathcal{Z}_k\left(\xi_1\xi_2\ldots\xi_k\right) = \xi_1\xi_2\ldots\xi_k + \xi_2\xi_3\ldots\xi_k\xi_1 + \ldots + \xi_k\xi_1\ldots\xi_{k-1} \qquad (57)$$

a solution of which is given by the *k-nary* relation:

$$\xi_1\xi_2\ldots\xi_k = q\,\xi_2\xi_3\ldots\xi_k\xi_1 \qquad (58)$$

i.e. under a cyclic permutation of the factors (of a product of k elements of the algebra) a factor q (k^th root of unity) appears.

From this relation it follows that:

- $(\xi_i)^k = 0$

- $\xi_1\xi_2\ldots\xi_k\xi_{k+1} = 0$

One also introduces $\bar{\xi}$'s duals to the ξ's as $k - 1$-grade elements obeying similar relations as the xi's but with q being replaced by $\bar{q} = q^{k-1}$. Furthermore the $\bar{\xi}$'s obey binary relations with the ξ's:

$$\xi_i\bar{\xi}_j = q\,\bar{\xi}_j\xi_i \, . \qquad (59)$$

Using then the same relations as in (48) (in fact a similar reasoning leads to interpret a^+ as $k - 1$ grade element and a as a 1-grade element).

One can check that the states:

$$|\xi\rangle_k = \exp_q(a^+\xi)|0\rangle \qquad (60)$$

are in fact eigenstates of the annihilation operator with ξ as an eigenvalue.

It is important to note that the GGCS constructed in this paper, are very general as on one hand they are related to a generalized deformation of the fermionic oscillator and it is achieved for any degree of nilpotency of the operators (and Grassmann variables used). As a matter of fact these results generalize the results obtained in [16] and [17].

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